

Home Work 2

กำหนด $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ และ

$$\begin{aligned} Cov[b_1, b_1, b_1] &= 0.05 \\ Cov[b_2, b_2, b_2] &= 0.025 \\ Cov[b_1, b_2, b_2] &= 0.001 \\ Cov[b_1, b_1, b_2] &= 0.001 \end{aligned}$$

Find $Cov[x \otimes xx^T]$

Solution

From $[A \otimes A][x \otimes xx^T]A^T = b \otimes bb^T \rightarrow [A \otimes A]Cov[x \otimes xx^T]A^T = Cov[b \otimes bb^T]$
 $\rightarrow Cov[x \otimes xx^T] = [A \otimes A]^{-1}Cov[b \otimes bb^T](A^{-1})^T$

$$[A \otimes A] = \begin{bmatrix} 1 \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} & 2 \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \\ 2 \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} & 5 \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 2 & 5 & 4 & 10 \\ 2 & 4 & 5 & 10 \\ 4 & 10 & 10 & 25 \end{bmatrix}$$

$$[A \otimes A]^{-1} = \begin{bmatrix} 25 & -10 & -10 & 4 \\ -10 & 5 & 4 & -2 \\ -10 & 4 & 5 & -2 \\ 4 & -2 & -2 & 1 \end{bmatrix}$$

$$(A^{-1})^T = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

$$Cov[b \otimes bb^T] = \begin{bmatrix} Cov[b_1, b_1, b_1] & Cov[b_1, b_1, b_2] \\ Cov[b_1, b_1, b_2] & Cov[b_1, b_2, b_2] \\ Cov[b_1, b_1, b_2] & Cov[b_1, b_2, b_2] \\ Cov[b_1, b_2, b_2] & Cov[b_2, b_2, b_2] \end{bmatrix} = \begin{bmatrix} 0.05 & 0.001 \\ 0.001 & -0.001 \\ 0.001 & -0.001 \\ -0.001 & 0.025 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow xx^T = \begin{bmatrix} x_1^2 & x_1x_2 \\ x_2x_1 & x_2^2 \end{bmatrix} \rightarrow [x \otimes xx^T] = \begin{bmatrix} x_1 \begin{bmatrix} x_1^2 & x_1x_2 \\ x_2x_1 & x_2^2 \end{bmatrix} \\ x_2 \begin{bmatrix} x_1^2 & x_1x_2 \\ x_2x_1 & x_2^2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} x_1^3 & x_1^2x_2 \\ x_1^2x_2 & x_2^2x_1 \\ x_1^2x_2 & x_2^2x_1 \\ x_2^2x_1 & x_2^3 \end{bmatrix}$$

$$\begin{aligned} \rightarrow Cov[x \otimes xx^T] &= \begin{bmatrix} Cov[x_1, x_1, x_1] & Cov[x_1, x_1, x_2] \\ Cov[x_1, x_1, x_2] & Cov[x_2, x_2, x_1] \\ Cov[x_1, x_1, x_2] & Cov[x_2, x_2, x_1] \\ Cov[x_2, x_2, x_1] & Cov[x_2, x_2, x_2] \end{bmatrix} \\ &= [A \otimes A]^{-1} Cov[b \otimes bb^T] (A^{-1})^T = \begin{bmatrix} 25 & -10 & -10 & 4 \\ -10 & 5 & 4 & -2 \\ -10 & 4 & 5 & -2 \\ 4 & -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0.05 & 0.001 \\ 0.001 & -0.001 \\ 0.001 & -0.001 \\ -0.001 & 0.025 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \end{aligned}$$

$$\therefore Cov[x \otimes xx^T] = \begin{bmatrix} Cov[x_1, x_1, x_1] = 5.84 & Cov[x_1, x_1, x_2] = -2.307 \\ Cov[x_1, x_1, x_2] = -2.307 & Cov[x_2, x_2, x_1] = 0.909 \\ Cov[x_1, x_1, x_2] = -2.307 & Cov[x_2, x_2, x_1] = 0.909 \\ Cov[x_2, x_2, x_1] = 0.909 & Cov[x_2, x_2, x_2] = -0.357 \end{bmatrix}$$

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